Outer billiard outside regular polygons

Nicolas Bedaride, Julien Cassaigne

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ABSTRACT

We consider the outer billiard outside a regular convex polygon. We deal with the case of regular polygons with $3, 4, 5, 6$ or 10 sides. We describe the symbolic dynamics of the map and compute the complexity of the language.

Keywords: Symbolic dynamic, outer billiard, complexity function, words.
AMS codes: 37A35 ; 37C35; 05A16; 28D20.

1 Introduction

An outer billiard table is a compact convex domain P. Pick a point M outside P. There are two half-lines starting from M tangent to P, choose one of them, say the one to the right from the view-point of M, and reflect M with respect to the tangency point. One obtains a new point, N, and the transformation $T : TM = N$ is the outer (a.k.a. dual) billiard map, see Figure 1. The map T is not defined if the support line has a segment in common with the outer billiard table. In the case where P is a convex polygon the tangency points are vertices of P. The set of points for which T or any of its iterations is not defined is contained in a countable union of lines and has zero measure. The dual billiard map was introduced by Neumann in [Neu59] as a toy model for planet orbits.

We are mainly interested in the case of polygons. A particular class of polygons has been introduced by Kolodziej et al. in several articles, see [VS87, Koł89, GS92]. This class is named the quasi-rational polygons and contains all the regular polygons. They prove that every orbit of the outer billiard outside a polygon in this class is bounded. Recently Schwartz described a family of polygons for which there exist unbounded orbits, see [Sch07, Sch09]. In the case of the regular pentagon Tabachnikov completely

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described the dynamics of the outer billiard map in terms of symbolic dynamics, see [Tab95b].

In this paper we consider the outer billiard map outside regular polygons, and analyze the symbolic dynamics attached to this map. We are interested in the cases where the polygon has $3, 4, 5, 6$ or 10 sides and give a complete description of the dynamics by a characterization of the language and the complexity function. We compute the global complexity of these maps, and generalize the result of Gutkin and Tabachnikov, see [GT06].

The description of a language associated to a dynamical system, is a way to understand the dynamics. Even if the dynamics is quite simple, the combinatorics of the language can be non trivial. Moreover the link between the combinatorics and the geometry is useful to see the properties of the dynamical system. For example in the case of inner billiard inside a square, the dynamics is easy since either the orbit is periodic or it is the orbit of a point under a rotation. But the computation of the complexity function via the bispecial words is not so simple, see [CHT02].

The study of the symbolic dynamics of these map outside a polygon is just at the beginning. By a result of Buzzi, see [Buz01], we know that the topological entropy is zero, thus the complexity growths as a sub-exponential. Recently Gutkin andTabachnikov proved that the complexity function of polygonal outer billiard is always bounded by a polynomial, see [GT06].

2 Overview of the paper

In Section 3 we define the outer billiard map outside a polygon. In Section 4 we recall the basic definitions of word combinatorics and explain the partition associated to the dual billiard map. In Section 5 we simplify our problem, then in Section 6 we recall the different results on the subject, and in Section 7 we can state our results. The main case is the pentagonal case. We give the proof for this case, the other cases can be treated by a similar analysis. First in Section 8 we recall some facts on piecewise isometries. Then in Section 9 we use an induction method to describe the dynamics of the dual billiard map. Finally in Sections 10, 11 and 12 we describe the language of the dual billiard map outside the regular pentagon and finish the proof. Section 13 finishes the paper with a similar result for the regular decagon.

3 Outer billiard

We refer to [Tab95a] or [GS92]. We consider a convex polygon P in $\mathbb{R}^2$ with k vertices. Let $\overline{P} = \mathbb{R}^2 \setminus P$ be the complement of P.
We fix an orientation of $\mathbb{R}^2$.

Definition 1. *Denote by σ_1 the union of straight lines containing the sides of P. We consider the central symmetries $s_i, i = 0 \dots k-1$ about the vertices of P. Define σ_n, where $n \geq 2$ is an integer, by $\sigma_n = \bigcup_{i=0}^{k-1} s_i \sigma_{n-1}$. Now the singular set is defined by:*

$$Y = \bigcup_{n=1}^{\infty} \sigma_n.$$

For any point $M \in \overline{P} \setminus Y$, there are two half-lines R, R' emanating from M and tangent to P, see Figure 1. Assume that the oriented angle R, R' has positive measure. Denote by A^+, A^- the tangent points on R respectively R'.

Definition 2. *The outer billiard map is the map T defined as follows:*

$$\begin{array}{cccc} T \;:\; & \overline{P} \setminus Y & \rightarrow & \overline{P} \setminus Y \\ & M & \mapsto & s_{A^+}(M) \end{array}$$

where s_{A^+} is the reflection about A^+.

Remark 1. *This map is not defined on the entire space. The map T^n can be defined on $\overline{P} \setminus \sigma_n$, but on this set T^{n+1} is not everywhere defined. The definition set $\overline{P} \setminus Y$ is of full measure in $\overline{P}$.*

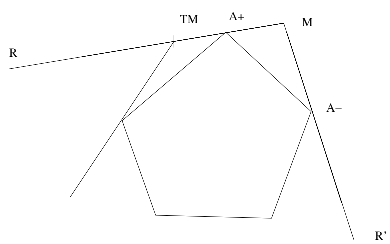

Figure 1: Outer billiard map

Two important families of polygons have been defined in the study of this map: the rational polygons and the quasi-rational polygons.

Definition 3. *A polygon P is said to be rational if the vertices of P are on a lattice of $\mathbb{R}^2$.*

The definition of the quasi-rational polygons is more technical and we will not need it here. We just mention the fact that every regular polygon is a quasi-rational polygon.

4 Symbolic dynamics

4.1 Definitions

4.1.1 Words

For the notions of word, factor, substitution we refer to [PF02]. In the following, we will deal with several infinite words, thus we need a general definition of a language.

Definition 4. *A language L is a sequence $(L_n)_{n\in\mathbb{N}}$ where L_n is a finite set of words of length n such that for any word $v \in L_n$ there exists two letters a, b such that av and vb are elements of L_{n+1}, and all factors of length n of elements of L_{n+1} are in L_n. $Fact(L)$ is the set of all factors of L.*

Definition 5. *The complexity function of a language L is the function $p : \mathbb{N} \to \mathbb{N}$ defined by $p(n) = card(L_n)$.*

4.1.2 Complexity

First we recall a result of the second author concerning combinatorics of words [Cas97].

Definition 6. *Let $L = (L_n)_{n\in\mathbb{N}}$ be a language. For any $n \geq 0$ let $s(n) := p(n+1) - p(n)$. For $v \in L_n$ let*

$$m_l(v) = card\{a \in \mathcal{A}, av \in L_{n+1}\},$$
$$m_r(v) = card\{b \in \mathcal{A}, vb \in L_{n+1}\},$$
$$m_b(v) = card\{(a, b) \in \mathcal{A}^2, avb \in L_{n+2}\}.$$

- *A word v is called right special if $m_r(v) \geq 2$.*
- *A word v is called left special if $m_l(v) \geq 2$.*
- *A word v is called bispecial if it is right and left special.*
- *BL_n denotes the set of bispecial words of length n.*
- *$b(n)$ denotes the sum $b(n) = \sum_{v\in BL_n} i(v)$, where $i(v) = m_b(v) - m_r(v) - m_l(v) + 1$.*

Lemma 1. *Let L be a language. Then the complexity of L satisfies for every integer $n \geq 0$:*

$$s(n+1) - s(n) = b(n).$$

For the proof of the lemma we refer to [Cas97] or [CHT02].

Definition 7. *A word v such that $i(v) < 0$ is called a weak bispecial. A word v such that $i(v) > 0$ is called a strong bispecial. A bispecial word v such that $i(v) = 0$ is called a neutral bispecial.*

4.2 Coding

We introduce a coding for the dual billiard map. Recall that the polygon P has k vertices.

Definition 8. *The sides of P can be extended in half-lines in the following way: denote these half lines by $(d_i)_{0\leq i\leq k-1}$ we assume that the angle (d_i, d_{i+1}) is positive. They form a cellular decomposition of $\overline{P}$ into k closed cones $V_0, \ldots, V_{k-1}$. By convention we assume that the half line d_i is between V_{i-1} and V_i see Figure 2.*

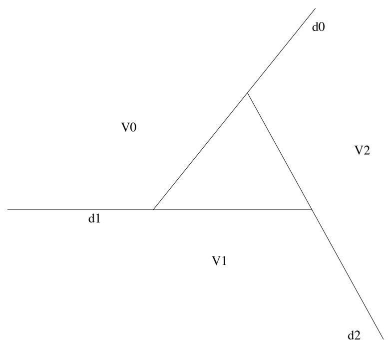

Figure 2: Partition

Now we define the coding map, we refer to Definition 1:

Definition 9. *Let ρ be the map*

$$\begin{array}{rcccc} \rho & : & \overline{P}\setminus Y & \rightarrow & \{0,\ldots,k-1\}^{\mathbb{N}} \\ & & M & \mapsto & (u_n)_{n\in\mathbb{N}} \end{array}$$

where $u_n = i$ if and only if $T^n M \in V_i$.

Now consider the factors of length n of u, and denote this set by $L_n(M)$. Remark that $T^n M \in V_i \cap V_{i+1}$ is impossible if $M \notin Y$.

Definition 10. *We introduce the following set*

$$L_n = \bigcup_{M\in\overline{P}\setminus Y} L_n(M).$$

This set corresponds to all the words of length n which code outer billiard orbits. Then the set $L = \bigcup_n L_n$ is a language. It is the language of the outer billiard map. We denote the complexity of L by $p(n)$, see Definition 5.

$$p(n) = card(L_n).$$

Definition 11. *The set $\{0,\ldots,k-1\}^{\mathbb{N}}$ has the natural product topology. Then X denotes the closed set $X = \overline{\rho(\overline{P}\setminus Y)}$.*

5 Simplification of the problem

In this part we introduce a new map $\hat{T}$ to simplify the problem. This map is not the first return map on V_0. We use it instead of the first return map, because it seems that this map can be used for all the regular polygons.

5.1 First remarks

Lemma 2. *Let P be a convex polygon, and h an affine map of $\mathbb{R}^2$ preserving the orientation, then the languages of the outer billiard maps outside P and $h(P)$ are the same.*

Proof. The proof is left to the reader. $\square$

Remark that an affine map preserves the set of lattices. Thus if P is rational, then $h(P)$ is rational for each affine map h. Also, the outer billiard map outside any triangle has the same language, so it is sufficiant to study the equilateral triangle, see Lemma 2.

5.2 A new coding for the regular polygon

Let P be a regular polygon with k vertices, and R be the rotation of angle $-2\pi/k$, centered at the center of the polygon. Consider one sector V_0 and define the map:

$$\begin{array}{ccc} \bigcup_i V_i & \to & \mathbb{N} \\ y & \mapsto & n_y \end{array}$$

where the integer n_y is the smallest integer which maps the sector V_i containing y to V_0 by a power of the rotation R. Then we define a new map

Definition 12. *The map $\hat{T}$ is defined in V_0 by the formula*

$$\hat{T}(x) = R^{n_{Tx}}Tx.$$

Lemma 3. *The integer n_{Tx} takes the values 1 to $j = \lfloor \frac{k+1}{2} \rfloor$. The map $\hat{T}$ is a piecewise isometry on j pieces.*

Proof. We will treat the case where k is even, the other case is similar. Assume $k = 2k'$, then we can assume that the regular polygon has as vertices the complex numbers $e^{i\pi n/k'}, n = 0 \dots k-1$ and that V_0 has 1 as vertex. Consider the cone $V = TV_0$ of vertex 1 obtained by a central symmetry from the cone V_0. We must count the number of intersection points of this cone with the cones $V_i, i = 0 \dots k-1$. The polygon is invariant by a central symmetry, this symmetry maps the cone V_i, to the cone $V_{k'+i}$. We deduce that if the cone V_i intersects V, then the cone $V_{k'-i}$ cannot intersect it. Moreover it is clear that each cone $V_i, 1 \leq i \leq k'$ intersects V, thus n_{T_x} takes the values 1 to k'. Now for each value of n_{Tx} we obtain an isometry, thus we obtain a piecewise map defined on $j = k'$ sets. $\square$

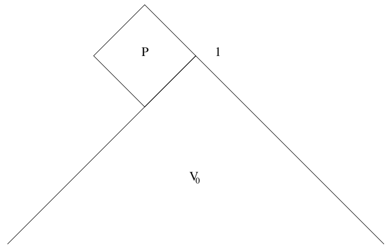

Figure 3: New coding

Definition 13. *Let η be the map defined as*

$$\begin{array}{cccc} \eta & : & V_0 \setminus Y & \rightarrow & \{1, \dots, j\}^{\mathbb{N}} \\ & & x & \mapsto & (n_{T(\hat{T}^i x)})_{i \in \mathbb{N}} \end{array}$$

As in Definition 10, let L' be the language of $\hat{T}$ related to the coding η.

Lemma 4. *We have*

$$\hat{T}^k(x) = R^{n_{T^k x}} T^k x.$$

Proof. This lemma is a consequence of the following fact: If A, B are two consecutive vertices of the polygon for the orientation, denote by s_A, s_B the central symmetries about them, then we have $Rs_B = s_A R$. This relation implies that R and T commute:

$$RT = TR.$$

□

Lemma 5. *If $x \in V_0 \setminus Y$, then the codings $(u_n) = \rho(x)$ and $(v_n) = \eta(x)$ are linked by*

$$v_n = u_{n+1} - u_n \quad mod \quad k.$$

Proof. Consider two consecutive elements of the sequence u with values a, b. It means $T^n x \in V_a, T^{n+1} x \in V_b$. Now the rotation R^{b-a} maps V_b to V_a, thus we deduce that $R^{b-a} T[T^n x]$ belongs to V_a, this implies with the help of preceding Lemma that $v_n = b - a \quad mod \quad k$. □

The preceding Lemma implies that the study of the map $\hat{T}$ will give information for T.

Lemma 6. *With the notations of Definitions 10, 13 we have*

$$p_L(n) = k p_{L'}(n-1).$$

Moreover the map $\begin{array}{ccc} L & \rightarrow & L' \\ u & \mapsto & v \end{array}$ *defined by $v_i = u_{i+1} - u_i$, for $0 \le i \le n-2$ if $u = u_0 u_1 \dots u_{n-1}$ is a k-to-one map.*

Proof. The regular polygon is invariant by the rotation R, thus the points x and $R^i x, 0 \leq i \leq k-1$ have the same coding in L'. Thus the map is not injective and the pre-image of a word consists of k word. By definition it is surjective. Now the formula for the complexity function is an obvious consequence of the formula $v_n = u_{n+1} - u_n$, see Lemma 5. □

6 Background

Few results are known about the complexity of the outer billiard map. Gutkin and Tabachnikov proved in [GT06] the following result.

Theorem 1. *Let P be a convex polygon*

- *If P is a regular polygon with k vertices then there exist $a, b > 0$ such that*

$$an \leq p(n) \leq bn^{r+2}.$$

  The integer r is the rank of the abelian group generated by translations in the sides of P. We have $r = \phi(k)$, where ϕ is the Euler function.

- *If P is a rational polygon, then there exist $a, b > 0$ such that*

$$an^2 \leq p(n) \leq bn^2.$$

In fact their theorem concerns the more general family of quasi-rational polygons that includes the regular ones, but we will not prove a result about this family of polygons. Remark that for the regular pentagon, we have $r = 4$.

7 Results

We obtain two types of results: The description of the language of the dynamics, and the computation of the complexity. The results are obtained for two types of polygons: the triangle, the square and the regular hexagon which are rational polygons; and the regular pentagon which is a quasi-rational polygon.

7.1 Languages

We characterize the languages of the outer billiard map outside regular polygons: We will use Lemma 6 and work with the language L'. Moreover we will give only infinite words. The language is the set of finite words which appear in the infinite words.

Definition 14. *Consider the three following endomorphisms of the free group F_3 defined on the alphabet $\{1;2;3\}$*

$$\sigma : \begin{cases} 1 \to 1121211 \\ 2 \to 111 \\ 3 \to 3 \end{cases} \qquad \psi : \begin{cases} 1 \to 2232232 \\ 2 \to 232 \\ 3 \to 2^{-1} \end{cases} \qquad \xi : \begin{cases} 1 \to 23222 \\ 2 \to 2 \\ 3 \to 3 \end{cases}$$

Theorem 2. *Let P be a triangle, a square, a regular hexagon or a regular pentagon. Then the language L' of the dynamics of $\hat{T}$ is the set of factors of the periodic words of the form z^ω for $z \in Z$, where*

- *If P is a triangle*

$$Z = \bigcup_{n\in\mathbb{N}} \{1(21)^n, 1(21)^n 1(21)^{n+1}\}.$$

- *If P is the square*

$$Z = \bigcup_{n\in\mathbb{N}} \{12^n\}.$$

- *If P is the regular hexagon*

$$Z = \bigcup_{n\in\mathbb{N}} \{23^n, 23^n 23^{n+1}\} \cup \{1\}.$$

- *If P is the regular pentagon then Z is the union of*

$$\bigcup_{n\in\mathbb{N}} \{\sigma^n(1), \sigma^n(12)\},$$

$$\bigcup_{n,m\in\mathbb{N}} \{\psi^m(2), \psi^m(2223), \psi^m \circ \sigma^n(1), \psi^m \circ \sigma^n(12)\},$$

$$\bigcup_{n,m\in\mathbb{N}} \{\psi^m \circ \xi \circ \sigma^n(1), \psi^m \circ \xi \circ \sigma^n(12)\}.$$

7.2 Complexity

In the statement of Theorem 3 we give the formula for $p_{L'}$. Lemma 6 can be used to obtain the formula for the complexity of the language L.

Theorem 3.

- *For a triangle, we have*

$$p_{L'}(n) = \frac{5n^2 + 14n + f(r)}{24},$$

where $r = n \mod 12$ and $f(r)$ is given by

r	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
f(r)	*24*	*29*	*24*	*9*	*8*	*21*	*24*	*17*	*0*	*-3*	*8*	*9*

- *For a square we obtain:*

$$p_{L'}(n) = \frac{1}{2}\lfloor\frac{(n+2)^2}{2}\rfloor.$$

- *For a regular hexagon:*

$$p_{L'}(n) = \lfloor\frac{5n^2+16n+15}{12}\rfloor.$$

- *For a regular pentagon, let β be the real number:*

$$\beta = \frac{14}{15} + \sum_{n\geq 0}\left(\frac{7}{48.6^n.2+14+2(-1)^n} + \frac{7}{18.6^n.2+14-(-1)^n}\right)$$
$$- \sum_{n\geq 0}\left(\frac{7}{78.6^n.2+14+5(-1)^n} + \frac{7}{48.6^n.2+14-5(-1)^n}\right).$$

  then we have

$$p_{L'}(n) \sim \frac{\beta n^2}{2},$$

  with $\beta \sim 1,060$.

- *For a regular decagon, there exists a constant $C > 0$ such that*

$$p_{L'}(n) \sim Cn^2.$$

7.3 Remarks

The proof uses the same method in all the cases. Thus we only treat the case of pentagon and decagon which are harder. The other cases are similar, the dynamics is quite elementary since all the orbits are periodic. The computation of the complexity function uses the same method as in the pentagonal case.

The differences between the cases $n = 3, 4, 6$ and $n = 5, 10$ come from the following facts. The first cases are much easier to study, for a dynamical point of view, because the respective polygons are on a lattice. This is not true anymore for $n = 5, 10$. In Figure 4 and 5 we show the different tilings of the plane obtained by the sequence $(T^{-n}d)_{n\in\mathbb{N}}$ where d is a line supporting an edge of the regular polygon. In the first cases, we obtain a regular tiling of the plane by regular polygons. In the case of a regular pentagon we see that the adherence of this orbit has a fractal structure. Remark that the case of the regular octogon has recently been studied by Schwartz, see [Sch10]. But the point of view is different, the paper focuses on the arithmetic graph and not on the symbolic dynamics.

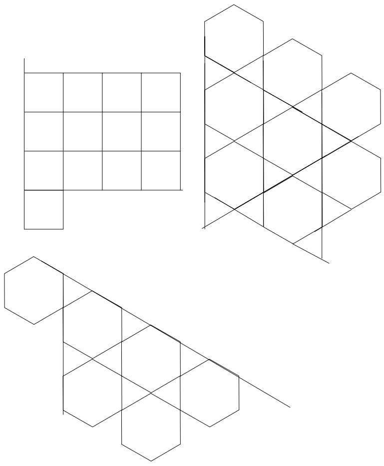

Figure 4: Regular polygon with $n = 3, 4, 6$

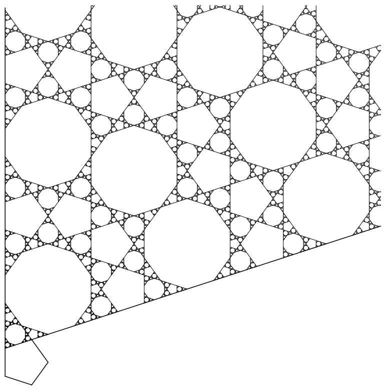

Figure 5: Regular pentagon

8 Piecewise isometry of Tabachnikov

In this section we recall some results proved by Tabachnikov in [Tab95b].

8.1 Definition and results

Consider Figure 6. We define a piecewise isometry (Z, G) on the union of two triangles

$$Z = AFC \cup HFE.$$

The two triangles are isoceles, the angle in A equals $2\pi/5$, $AF = 1$, and $AC = \frac{1+\sqrt{5}}{2}$. The map $G : Z \mapsto Z$ is defined as follows:

- a rotation of center O_1 and angle $-3\pi/5$ which sends C to E, on AFC.

- a rotation of center O_2 and angle $-\pi/5$ which sends H to C on HFE.

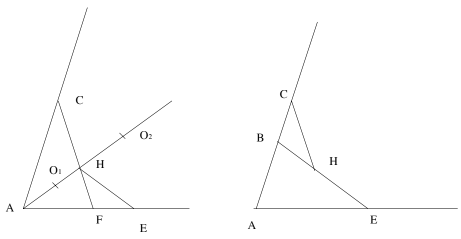

Figure 6: Piecewise isometry G

Definition 15. *Let σ be the substitution:*

$$\sigma : \begin{cases} 1 \to 1121211 \\ 2 \to 111 \end{cases}$$

and let u be its fixed point.

Definition 16. *We denote by V_{per} the set of periodic points for G , and by V_∞ the set $Z \setminus V_{per}$.*

Theorem 4. *[Tab95b] We have:*

1. *If x is a point with non periodic orbit under G, then the dynamical system $(O(x), G)$ is conjugated to $(O(u), S)$ where S is the shift map, and $O(x)$ denotes the closure of the orbit of x.*

2. *A connected component of V_{per} is a regular pentagon or a regular decagon.*

3. *Each point in a regular decagon has for coding an infinite word included in the shift orbit of $(\sigma^n(1))^\omega, n \in \mathbb{N}$. The points inside regular pentagons correspond to the words $(\sigma^n(12))^\omega, n \in \mathbb{N}$.*

Corollary 1. *The aperiodic points have codings included in the orbit $O(u)$.*

8.2 Link between (Z, G) and the outer billiard outside the regular pentagon

We will make more precise the statement of Lemma 3. We use the same definitions, but the sector will be denoted by V.

Definition 17. *The points refer to Figure 7. We define three sets*

- *U_1 is the triangle AEB.*

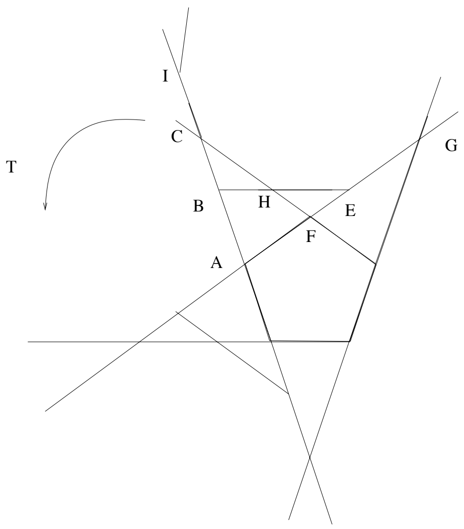

Figure 7: Definition of $\hat{T}$

- *U_2 is an infinite polygon with vertices IBE.*
- *U_3 is a cone of vertex I.*

A straightforward analysis of Figure 7 allows us to prove the following lemma.

Lemma 7. *The map $\hat{T}$, see definition 12, is defined on three subsets U_1, U_2, U_3.*

$$V = U_1 \cup U_2 \cup U_3.$$

The images of U_1, U_2, U_3 by $\hat{T}$ verify the following properties:

- *The first cell U_1 has for image the triangle ACF and $\hat{T}_{|U_1} = RT$.*
- *On the second cell U_2, we have $\hat{T}_{|U_2} = R^2T$, and the image of U_2 is an infinite polygon with vertices CFG.*
- *On the third set, we have $\hat{T}_{|U_3} = R^3T$, and the image of U_3 is a cone of vertex G.*
- *The union of the triangles ACF and HFE is invariant by $\hat{T}$:*

$$U_1 \cup (\hat{T}U_1 \cap U_2).$$

- *The restriction of the map $\hat{T}$ to this invariant set is the map (Z, G^{-1}).*
- *The set $[U_2 \setminus (\hat{T}U_1 \cap U_2)] \cup U_3$ is also invariant.*

9 Dynamics on $U_2 \cup U_3$

The last point of the preceding lemma implies that we can restrict our study to a piece of $U_2 \cup U_3$. By calculation with complex numbers we prove:

Lemma 8. *The dynamics on the invariant set* $[U_2 \setminus (\hat{T}U_1 \cap U_2)] \cup U_3$ *is given by:*

- *The map restricted to* U_3 *is a rotation by angle* $-\pi/5$.
- *The map restricted to* $U_2 \setminus Z$ *is a rotation by angle* $\pi/5$.

We use the coding related to $\hat{T}$, see Subsection 5.2. Using Lemma 7 we see that on the invariant set Z we have the same coding as in the piecewise isometry of Tabachnikov.

Definition 18. *We define the map*

$$\begin{array}{cccc} F & : & \{1;2;3\}^* & \rightarrow & Isom(\mathbb{R}^2) \\ & & v & \mapsto & F(v) \end{array}$$

where $v = v_0 \dots v_{n-1}$ *is a finite word over the alphabet* $\{1,2,3\}$, $F(v)$ *is the composition of isometries:* $F(v) = F(v_{n-1}) \circ \dots F(v_0)$. *The map* $F(i)$ *coincides with* $\hat{T}$ *on* U_i *for* $i = 1 \dots 3$.

Remark that $F(2)$ and $F(3)$ are rotations of opposite angles. $F(2)$ has angle $\pi/5$ and $F(1)$ has angle equal to $3\pi/5$.

Now some computations on isometries allow us to prove the following fact:

Lemma 9. *There exists a translation* t *and a rotation* u *such that*

- *For any word* v *of the language of* $\hat{T}$ *we have:*

$$F(\psi(v)) \circ t = t \circ F(v), F(\xi(v)) \circ u = u \circ F(v).$$

- *For the language of the map* $\hat{T}$*, we have equivalence between*
  - v *is the code of a periodic point.*
  - $\psi(v)$ *is the code of a periodic point.*
- *For the language of the map* $\hat{T}$*, we have equivalence between*
  - v *is a periodic word.*
  - $\xi(v)$ *is a periodic word.*

10 Proof of Theorem 2 for the regular pentagon

Before the proof we explain the statement of the theorem from a geometric point of view.

We can split the language in different sets of periodic words with following periods:

- The words given by iteration of σ on 1 or 12.
- The words obtained by the iterations of 2223 or 2 under ψ.
- The iterates of ψ on the periodic words of the first class. They corresponds to coding of orbits of the following points: Take one point in Z, translate it by some power of t.
- The words obtained by composition of ξ and a power of ψ on the first words.

We use Corollary 9. We construct three invariant regions which will glue together and form a fundamental domain for the action of the translation t. The three sets are given by

- The properties of the substitution ψ imply the existence of a translation t which is parallel to a side of the cone. Let j be an integer, then the set $Z + jt$ is not invariant by $\hat{T}$. Now consider its orbit:

  $$\bigcup_{i \in \mathbb{N}} \hat{T}^i(Z + jt).$$

  First by Lemma 8 we know that its orbit is inside $U_2 \cup U_3$. Z is made of one big triangle and one small triangle. We claim that the five first iterations form an invariant set made of eight big triangles and three small ones. The symbolic dynamics inside this set is given by the composition of ψ^j and σ. It is the first invariant ring.

- Now we look at the second substitution ξ. Corollary 9 shows that there exists an invariant ring corresponding to the cells of the orbit under the shift of $(32222)^\omega$. The symbolic dynamics inside this set is given by the composition of ξ and one iterate of σ.
- Moreover by Lemma 8 there is an invariant polygon inside U_2, which is a regular decagon.

These three invariant rings glue together by trigonometric arguments.

Now we have a compact set inside V with a symbolic description. These words are thus obtained as the $\mathbb{Z}^2$ sequences:

$$\psi^j \circ \sigma^i(1), \psi^j \circ \xi \circ \sigma^i(1), \psi^j(2223).$$

Now consider the parallelogram in V with sides parallel to the axis of V and of lengths $|t|$ and the golden mean. Then consider the images of it by all the power of t. It forms a strip. Let m be a point in V, then either it is in the strip or there exists an integer n_0 such that $\hat{T}^{n_0}m$ is inside this strip. Indeed outside the strip $\hat{T}$ acts as a rotation. Thus the dynamics of every point can be described with the preceding description.

11 Bispecial words

In this Section we will describe the bispecial words of the language of the outer billiard map outside the regular pentagon.

Definition 19. *We introduce different maps and words to simplify the statement of the result.*

- $\Phi \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 23 \end{cases}$
- $\tilde{\psi} \begin{cases} 1 \mapsto 23232 \\ 2 \mapsto 32 \\ 3 \mapsto 3 \end{cases}$
- $\tilde{\xi} \begin{cases} 1 \mapsto 3222 \\ 2 \mapsto 2 \end{cases}$
- $\tilde{\psi}_{|\{1,2\}^*} = \tilde{\beta}$.
- $\tilde{\psi}_{|\{2,3\}^*} = \tilde{\chi}$.
- $\hat{\tilde{\chi}}(w) = \tilde{\chi}(w)3$ *for all word* w.
- $\hat{\tilde{\xi}}(w) = 222\tilde{\xi}(w)$ *for all word* w.
- $\hat{\sigma}(w) = 11\sigma(w)11$ *for all word* w.
- $\hat{\Phi}(w) = \Phi(w)2$ *for all word* $w \in \{2,3\}^*$.
- $\hat{\tilde{\beta}}(w) = 23232\tilde{\beta}(w)$ *for all word* w.
- $x_n = \hat{\sigma}^n(1)$ *for all integer* n.
- $y_n = \hat{\sigma}^n(1111)$ *for all integer* n.
- $z_n = \hat{\sigma}^n(12121)$ *for all integer* n.
- $t_n = \hat{\sigma}^n(1^7)$ *for all integer* n.

Remark that we have $\psi = \Phi \circ \tilde{\psi} \circ \Phi^{-1}, \xi = \Phi \circ \tilde{\xi} \circ \Phi^{-1}$.

The aim of this part is to prove

Proposition 1. *The bispecial words of the language L' of the outer billiard outside the regular pentagon form 24 families, according to preceding definition.*

- *The empty word* ε, *with* $i(\varepsilon) = 2$.
- *The word* 2, *with* $i(2) = 0$.
- *The weak bispecial words are with* $k, n \in \mathbb{N}$*:*

$$\hat{\Phi}(\hat{\tilde{\chi}}^k(2222)), \hat{\Phi}(\hat{\tilde{\chi}}^k(22322)), \hat{\Phi}(\hat{\tilde{\chi}}^k(232232)), \hat{\Phi}(\hat{\tilde{\chi}}^k(2323232)),$$

$$\hat{\Phi}(\hat{\tilde{\chi}}^k \circ \hat{\tilde{\xi}}(z_n)), \hat{\Phi}(\hat{\tilde{\chi}}^k \circ \hat{\tilde{\xi}}(t_n)), \hat{\Phi}(\hat{\tilde{\chi}}^k \circ \hat{\tilde{\beta}}(z_n)), \hat{\Phi}(\hat{\tilde{\chi}}^k \circ \hat{\tilde{\beta}}((t_n)),$$

$$z_n, t_n$$

$$\begin{tikzcd}
(12)^\omega \arrow[d] & 1^\omega \arrow[dl] \\
L_0 \arrow[loop left, "\sigma"] \arrow[d, "\tilde{\xi}"'] \arrow[dr, "\tilde{\beta}"] & \\
L_1 \arrow[d, "\tilde{\chi}"'] & L_2 \arrow[dl, "\tilde{\chi}"'] \\
L_4 \arrow[loop right, "\tilde{\chi}"] & \\
L_3 \arrow[u, "\tilde{\chi}"'] &
\end{tikzcd}$$

Figure 8: Construction of L_Φ

- *The strong bispecial words with* $k, n \in \mathbb{N}$*:*

$$\hat{\Phi}(\hat{\tilde{\chi}}^k(2)), \hat{\Phi}(\hat{\tilde{\chi}}^k(22)), \hat{\Phi}(\hat{\tilde{\chi}}^k(222)), \hat{\Phi}(\hat{\tilde{\chi}}^k(232)), \hat{\Phi}(\hat{\tilde{\chi}}^k(23232)), \hat{\Phi}(\hat{\tilde{\chi}}^k(3)),$$

$$\hat{\Phi}(\hat{\tilde{\chi}}^k \circ \hat{\tilde{\xi}}(x_n)), \hat{\Phi}(\hat{\tilde{\chi}}^k \circ \hat{\tilde{\xi}}(y_n)), \hat{\Phi}(\hat{\tilde{\chi}}^k \circ \hat{\tilde{\beta}}(x_n)), \hat{\Phi}(\hat{\tilde{\chi}}^k \circ \hat{\tilde{\beta}}(y_n)),$$

$$x_n, y_n$$

11.1 Notations

We use Figure 8 and define five languages L_0, L_1, L_2, L_4, L_3.

Definition 20. *We denote L_0, L_1, L_2, L_4, L_3 the languages made respectiveley by factors of the following words:*

- $L_0 = \bigcup_{n \geq 0} \sigma^n(1)^\omega \cup \sigma^n(12)^\omega$.
- $L_1 = \tilde{\xi}(L_0)$.
- $L_2 = \tilde{\beta}(L_0)$.
- $L_3 = \bigcup_{m \geq 1} \tilde{\chi}^m(L_3) \cup \tilde{\chi}^m(L_2) \cup \tilde{\chi}^m(L_1)$.
- $L_4 = 2^\omega \cup (223)^\omega$.

11.2 Step one

11.2.1 Simplification of the problem

Lemma 10. *The language L' of the outer billiard map outside a regular pentagon is the union:*

$$L' = Fact(\Phi(L_0 \cup L_1 \cup L_2 \cup L_4 \cup L_3)).$$

Proof. By using the conjugation by Φ and Theorem 2 we obtain a description of our language. The map Φ^{-1} is given by $\begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 2^{-1}3 \end{cases}$. Thus we deduce that $\Phi^{-1}(L_0) = L_0$, and deduce the result. □

Definition 21. *By preceding Lemma L' is the union of L_0 and the image by Φ of a set. We denote $L_\Phi = L_1 \cup L_2 \cup L_4 \cup L_3$ so that $L' = \Phi(L_\Phi) \cup L_0$.*

The preceding operations can be summarized in Figure 8.

11.2.2 Study of the map Φ

In this part we explain how to manage the map Φ and restrict the study to the language L_Φ. In order to prove this result we use a synchronization lemma

Lemma 11. *If w is a factor of the language $\Phi(L_\Phi)$, then there exists a unique triple (s, v, p) such that $w = s\Phi(v)p$ with $s \in \{\varepsilon, 3\}, v \in L_\Phi, p \in \{\varepsilon, 2\}$, with the additional condition $v \in \mathcal{A}^*2 \Rightarrow p = 2$.*

Proof. First we are only interested in the case where $w = s\Phi(v)$. If w begins by 1, then it is clear that w can be written in the form $\Phi(1v)$. If w begins by 2 then three possibilities appear for the beginning: 21 thus we write $w = \Phi(21v)$; or 22 and w can be written as $\Phi(2v)$; and last possibility 23, then $w = \Phi(3v)$. If w begins by 3, we do the same thing and remark that the only problem is if w begins with 32. In this case we have $s = 3$. The first part of the lemma is proven. Now we consider words of the form $w = \Phi(v)p$. The proof is similar: the uniqueness is a consequence of the proof. □

Corollary 2. *A word w is bispecial in the language L' if and only if either*

- 1 *occurs in w and w is bispecial in L_0, with same index.*
- 3 *or* 22 *occurs un w, and $w = \hat{\Phi}(v)$ with v bispecial in L_Φ*
- $w = \varepsilon$,
- $w = 2$.

Proof. We apply preceding Lemma to w. Now if w is a bispecial word of $\Phi(L_\Phi)$, then $s=\varepsilon$, indeed the properties of Φ imply $13, 33$ do not belong to $\Phi(L_\Phi)$. Since the image of each letter by Φ ends with the same letter we remark that $xw \in \Phi(L_\Phi) \Rightarrow xv \in L_\Phi$. There are two cases

- $p=\varepsilon$: then the extensions of w must belong to $\Phi(L_\Phi)$. It implies $v1 \in L_\Phi$ and either $v2$ or $v3$ belong to L_Φ (or both).
- If $p=2$, then $w1$ or $w2$ or $w3$ belong to $\Phi(L_\Phi)$. It implies $v21 \in L_\Phi$ resp. $v22 \in L_\Phi$ or $v23 \in L_\Phi$ resp. $v3 \in L_\Phi$.

Then we can summarize this study in the four cases

- $w1, w2 \in \Phi(L_\Phi)$, then $v21$ and $v22 \quad$ or $\quad v23 \in L_\Phi$. So $v2$ is bispecial in L_Φ.
- $w1, w3 \in \Phi(L_\Phi)$, then $v2, v3 \in L_\Phi$, so v is bispecial in L_Φ.
- $w2, w3 \in \Phi(L_\Phi)$, then $v2, v3 \in L_\Phi$, so v is bispecial in L_Φ.
- $w1, w2, w3 \in \Phi(L_\Phi)$, then both v and $v2$ are bispecial in L_Φ.

This finishes the proof. □

Remark 2. *This lemma allows us to forget the map Φ until Section 12 and to study the bispecial words of the language L_Φ.*

11.3 Abstract of the method

The method to list the bispecial words is the following. We begin by the bispecial words which are not in the intersection of two of the languages L_1, L_2, L_4, L_3. For the language L_4 we prove that these words are images of bispecial words of $L_2 \cup L_1$. Then we prove that bispecial words in $L_1 \cup L_2$ are images of bispecial words in L_0, finally we list the bispecial words of L_0. Then it remains to treat the words which are in the intersection of two languages.

11.4 Different languages

We will need the following result.

Lemma 12. *We have*

$$L_0 \subset \{1, 12\}^*, L_1 \subset \{3222, 32222\}^*, L_2 \subset \{23, 223\}^*,$$

$$L_4 \subset \{3, 32\}^*, L_3 \subset \{2, 223\}^*$$

Proof. The proof just consists in the remark that 22 does not appear in L_0. Thus in L_2, the word 32 appears if u contains 1 and 322 appears when u contains 21. For L_1, the word 3222 appears in $\xi(1)$, and the word 32222 appears from the word 12. From the image by $\tilde{\chi}$ of $32, 3222$ and 32222 we deduce the result. □

11.4.1 Language L_4

Proposition 2. *If w is a non empty bispecial word of L_4, then we have $w = s\tilde{\chi}(v)p$ with $s = \varepsilon, p = 3$, where $v \in L_\Phi$ is a bispecial word of L_Φ with the same extansions as w.*

Proof. By the definition of L_4 we have $w \in F(\chi(L_1 \cup \dots L_3))$. By Lemma 12 a bispecial word must begin and end with the letter 3. Now since L_4 is built from the words 3 and 32 we can remark that these words are images of 3 and 2 by $\tilde{\chi}$. The last letter of w can not be the image by $\tilde{\chi}$ of 3 since it must be prolonged by 2. □

Corollary 3. *If w is a bispecial word of L_4 then there exists an integer k such that $w = \hat{\tilde{\chi}}^k(v)$, with $v \in L_1 \cup L_2 \cup L_3$. Moreover v is a bispecial word in L_Φ with the same multiplicity.*

Proof. By the preceding result $w = \hat{\tilde{\chi}}(v)$ and v is a bispecial word. Thus if v is not empty, we deduce $w = \hat{\tilde{\psi}}^2(v')$. We do the same thing until v' belongs to $L_1 \cup L_2 \cup L_3$. This must happen since the lengths decrease. □

11.4.2 Language L_3

Lemma 13. *The bispecial words in this language are:*

$$\{\varepsilon, 2, 22\}.$$

Proof. The proof is left to the reader. □

11.4.3 Language L_1

Lemma 14. *A bispecial word w of L_1 fulfills one of the following facts.*

- $w \in \{\varepsilon, 2, 22, 222, 2222, 22322\}$.
- $w = \hat{\tilde{\xi}}(v)$ *where v is bispecial in L_0, $v \neq \varepsilon$.*

Proof. First it is clear that a bispecial word in L_1 must begin and end with 2. If w is not a factor of 232232, then assume w does not contains 222 as a factor. By Lemma 12, $L_1 \subset \{2, 322\}^*$ we deduce that w is a factor of $(223)^\omega$. Thus we have $w = (223)^k 22$, and we deduce $k = 1$ since $w \notin F(232232)$. There remains one case corresponding to $222 \in F(w)$. Then by definition of L_1 we have $w \in \tilde{\xi}(L_0)$. By Lemma 12 the number of consecutive 2 is equal to 3 or 4, moreover the letter 3 is isolated. Then if w is a bispecial word we deduce that w begins with 2. Now $2w$ must be a word of the language. We deduce that w begins with three letters 2. Now we remark that 3222 is equal to $\hat{\xi}(1)$. This implies that the suffixe of w which prolong 222 is the image of one word by $\hat{\xi}$. To finish the proof it remains to list the bispecial words factors of 232232. □

11.4.4 Language L_2

Lemma 15. *A bispecial word w of L_2 fulfills one of the following facts.*

- $w \in F(23232)$.
- $w = \hat{\hat{\beta}}(v)$ *where v is bispecial in L_0, $v \neq \varepsilon$.*

Proof. First it is clear that a bispecial word in L_1 must begin and end by 2. By Lemma 12, we obtain $L_2 \subset \{23232, 2323232\}^*$. We split the proof in two cases: $23232 \in F(w)$ or $23232 \notin F(w)$. For the first case since w is a bispecial word we have $2w \in L_2$. Then Lemma 12 shows that 22 can only be extended by 3232, thus w begins with 23232, and $w = \hat{\hat{\beta}}(v)$. $\square$

11.4.5 Language L_0

Lemma 16. *The language L_0 fulfills the following three properties:*

- $11211 \notin L_0$.
- $22 \notin L_0$.
- *Three consecutive occurences of* 2 *are of the forms:* 21^l212 *or* 2121^l2 *with* $l \in \{1, 4; 7\}$.

The followings words are bispecial words of L_0:

- 1^i *for* $i = 0, 2, 3, 5, 6$ *and it is an ordinary bispecial word:* $i(1^i) = 0$. *Thus they do not modify the complexity.*
- 1^i *for* $i = 1, 4$ *and it is a strong bispecial word:* $i(1) = i(1^4) = 1$.
- $12121, 1^7$ *are weak bispecial words:* $i(12121) = i(1^7) = -1$.

The proof is left to the reader.

Lemma 17. *We have different cases for a non-ordinary bispecial word w of L_0.*

- $w = 1^n, n \in \{1, 4, 7\}$.
- $w = 12121$.
- $w = \hat{\sigma}(v)$ *where v is a non-ordinary bispecial word of L_0 and* $i(w) = i(v)$.

Proof. First we consider the words without 2, then the word is a power of 1, and preceding Lemma shows the different possibilities. Now if the word contains only one letter 2, then the word has the form $w = 1^m21^n$, now the fact that $11211 \notin L_0$ (see preceding Lemma) shows that the only possibility is 121 which is ordinary. The case where w contains at least two letters 2 remains. Then either w contains 212 or the word is a factor of $\sigma(L_0)$. We have different subcases for a bispecial word $w = 1^m 2 \dots 21^n$ of L_0 factor of u due to the preceding Lemma.

- $m = 1$, then the word $1w = 112 \dots 21^n$ must belong to the language. This implies that $w = 1212 \dots 21^n$, but the fact that $2w$ exists implies now that $w = 12121$.
- $m \in \{4, 7\}$, then $m = 7$ is cleary impossible. One case remains which can be written by symmetry as $w = 11112 \dots 21111$. An easy argument of synchronization finishes the proof.

□

The preceding lemma implies that the bispecial words of L_0 are of the form

Corollary 4. *For the long bispecial words there are four famillies of words.*

- $x_n = \hat{\sigma}^n(1), n \in \mathbb{N}$, $i(\hat{\sigma}^n(1)) = 1$.
- $y_n = \hat{\sigma}^n(1^4), n \in \mathbb{N}$, $i(\hat{\sigma}^n(1^4)) = 1$.
- $z_n = \hat{\sigma}^n(12121), n \in \mathbb{N}$, $i(\hat{\sigma}^n(12121)) = -1$.
- $t_n = \hat{\sigma}^n(1^7), n \in \mathbb{N}$, $i(\hat{\sigma}^n(1^7)) = -1$.

The two first famillies are made of strong bispecial words, the two last are weak bispecial words.

11.5 Intersection of languages

We interest in the words which belong to different languages.

In the following Lemma we denote by L_{ijk} the language intersection of the languages L_i, L_j and L_k for $i, j \in \{1 \dots 4\}$.

Lemma 18. *The words which belong to at least two languages are:*

- $22 \in L_{124}, 23 \in L_{1234}, 32 \in L_{1234}$.
- $222 \in L_{14}, 223 \in L_{124}, 232 \in L_{1234}, 322 \in L_{124}, 323 \in L_{23}$
- $2222 \in L_{14}, 2232 \in L_{124}, 2322 \in L_{124}, 2323 \in L_{23}, 3223 \in L_{24}, 3232 \in L_{23}$.
- $22322 \in L_{14}, 23223 \in L_{24}, 23232 \in L_{23}, 32232 \in L_{24}, 32323 \in L_{23}$.
- $232232 \in L_{24}, 232323 \in L_{23}, 323232 \in L_{23}$.

- $2323232 \in L_{23}, 3232323$.

Proof. We consider the words by family of different lengths. When we have listed all the words of a given length i, we consider the words of length $i+1$ which contain one of the preceding words as prefix or sufix. Then we use Lemma 12 to verify if this word is in two languages. This allows us to obtain the first list, after this it remains to look at the bispecial words. $\square$

Corollary 5. *We have*

$$L_1 \cap L_0 = \{\varepsilon, 2\}.$$

$$L_1 \cap L_4 = F(232).$$

$$L_1 \cap L_3 = F(22322) \cup \{222, 2222\}$$

$$L_4 \cap L_3 = F(232).$$

$$L_2 \cap L_3 = F(232232).$$

$$L_2 \cap L_4 = F(2323232)$$

Proof. The proof is left to the reader. $\square$

Corollary 6. *The bispecial words of L_Φ belonging to at least two of the languages L_1, L_2, L_3, L_4 are*

- *The ten strong bispecial words,*

$$\varepsilon, 2, 3, 22, 33, 222, 232, 323, 23232, 32323.$$

- *The four weak bispecial words :*

$$2222, 22322, 232232, 2323232.$$

11.6 Proof of Proposition 1

First, Lemma 11 implies that a bispecial word $w \in L'$ can be written as $w = \hat{\Phi}(v)$ where $v \in L_\Phi$ is a bispecial word. Now we are interested in a bispecial word v in L_Φ. Several cases appear

- If $v \in L_0$, then Corollary 4 shows that v is inside four famillies of words.
- If $v \in L_1$ then Lemma 14 implies that $v = \hat{\hat{\xi}}(v')$ with $v' \in L_0$ or v is element of a finite familly. Thus the preceding point completes the list of bispecial words of L_1.
- If $v \in L_2$, then Lemma 15 implies that except for a finite list of words, we can write $v = \hat{\hat{\beta}}(v')$ with $v' \in L_0$.

- If $v \in L_3$ then Lemma 13 gives the complete list of bispecial words.
- If $v \in L_4$, then by Corollary 3 we know that $v = \hat{\hat{\psi}}^k(v')$ where $v' \in L_1 \cup L_2 \cup L_3$.
- Corollary 6 and the preceding points allow us to finish the proof.

12 Proof of Theorem 3

We use Proposition 1 to compute the different lengths of these bispecial words. The proof is a calculation using linear algebra, thus we omit it.

Proposition 3. *The lengths of the bispecial words of the language L' are of the following form, with $n, k \in \mathbb{N} \cup \{0\}$:*

- *The lengths of the weak bispecial words is of the form*

$$\begin{cases} 10k+5 \\ 10k+7 \\ 10k+9 \\ 10k+11 \\ \frac{48.6^n(20k+24)+14(10k+7)-25(-1)^n(2k+1)}{35} \\ \frac{78.6^n(20k+16)+14(10k+3)+25(-1)^n(2k+1)}{35} \\ \frac{48.6^n(20k+16)+14(10k+3)-25(-1)^n(2k+3)}{35} \\ \frac{78.6^n(20k+24)+14(10k+7)+25(-1)^n(2k+3)}{35} \\ \frac{192.6^n+25(-1)^n-42}{35} \\ \frac{312.6^n-25(-1)^n-42}{35} \end{cases}$$

- *The lengths of the strong bispecial words is of the form*

$$\begin{cases} 4k+2 \\ 6k+3 \\ 8k+4 \\ 6k+5 \\ 8k+8 \\ 2k+3 \\ \frac{18.6^n(20k+24)+14(10k+7)-5(-1)^n(2k+1)}{35} \\ \frac{48.6^n(20k+24)+14(10k+7)+10(-1)^n(2k+1)}{35} \\ \frac{18.6^n(20k+16)+14(10k+3)-5(-1)^n(2k+3)}{35} \\ \frac{48.6^n(20k+16)+14(10k+3)+10(-1)^n(2k+3)}{35} \\ \frac{4.18.6^n-42+5(-1)^n}{35} \\ \frac{4.8.6^n-42+10(-1)^n}{35} \end{cases}$$

Lemma 19. *There exists $\beta > 0$ such that*

$$\sum_{i=0}^{N} b(i) \sim \beta N.$$

Moreover we can give a formula for β:

$$\begin{aligned}\beta =& \frac{14}{15} + \\ & \sum_{n\geq 0} \left(\frac{7}{48.6^n.2 + 14 + 2(-1)^n} + \frac{7}{18.6^n.2 + 14 - (-1)^n}\right) \\ & - \sum_{n\geq 0} \left(\frac{7}{78.6^n.2 + 14 + 5(-1)^n} + \frac{7}{48.6^n.2 + 14 - 5(-1)^n}\right).\end{aligned}$$

Corollary 7. *We deduce that $p(n) \sim \frac{\beta n^2}{2}$.*

Proof. The proof is a direct consequence of Lemma 1 and Lemma 19. □

13 Regular decagon

In this short section we explain how to deal with the case of the regular decagon. In fact this case can be deduced easily from the case of the regular pentagon. In Figure 9 the lengths are not correct, but the angles have correct values. We have drawn the partition and its image by $\hat{T}$.

13.1 Induction

Lemma 20. *The map $\hat{T}_{deca}$ is defined on five sets. The definitions of these sets are the following, see Figure 9 :*

- *The first set V_1 is a triangle, and $\hat{T}_{deca}$ is a rotation of angle $4\pi/5$ on this set.*
- *The second set V_2 is a quadrilateral, and $\hat{T}_{deca}$ is a rotation of angle $3\pi/5$ on this set.*
- *The third set is V_3 a quadrilateral, and $\hat{T}_{deca}$ is a rotation of angle $2\pi/5$ on this set.*
- *The fourth set V_4 is an infinite polygon with four edges, and $\hat{T}_{deca}$ is a rotation of angle $\pi/5$ on this set.*
- *The last one V_5 is a cone, and $\hat{T}_{deca}$ is a translation on this set.*

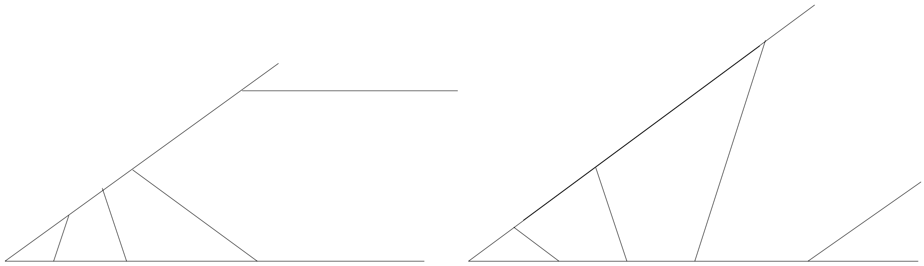

Figure 9: Definition of $\hat{T}$ for the decagon

Lemma 21. *Consider the maps $\hat{T}_{penta}, \hat{T}_{deca}$ related to the outer billiard map outside the regular pentagon respectively the regular decagon. Then consider the induced map on U_3, and denote it by $\hat{T}_{penta,3}$. Then there exists a translation s such that*

$$\hat{T}_{deca} = s^{-1} \circ \hat{T}_{penta,3} \circ s.$$

The results follow from Lemma 21.

Corollary 8. *There exists a bijective map θ between the coding of the decagon and the pentagon which is*

$$\begin{array}{cccc} \theta & : & \mathcal{L}'_{deca} & \rightarrow & \mathcal{L}'_{penta} \cap \rho(U_3) \\ & & \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{cases} & \rightarrow & \begin{cases} 322222 \\ 32222 \\ 3222 \\ 322 \\ 32 \end{cases} \end{array}$$

We use the same method as for the pentagon, and we deduce the language of the outer billiard map outside the regular decagon, and the complexity function.

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